Math 245B Lecture 10 Notes

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1 Universal Spaces

1.1 Embeddings into generalized cubes

In this lecture, $I = [0, 1]$.

Definition 1.1. A **generalized cube** is I^A for some $A \neq \varnothing$, with the product topology.

Definition 1.2. Let X be a topological space. The family $\mathcal{F} \subseteq C(X, I)$ **separates points and closed sets** if for all closed $E \subseteq E$ and $x \in E^c$, there is some $f \in \mathcal{F}$ such that $f(x) \notin \overline{f(E)}$.

The existence of such functions in a T_4 space is given by Tietze's extension theorem.

Definition 1.3. If $\mathcal{F} \subseteq C(X, I)$ separates points and closed sets, then tere exists $\mathcal{G} \subseteq C(X, I)$ such that for all closed $E \subseteq X$ and $x \in E^c$, there exists some $g \in \mathcal{G}$ such that $g(x) = 1$ and $g|_E = 0$.

Proof. For all x, E as above, choose f which separates them; that is, $f(x) \notin \overline{f(E)}$. Then x is contained in an interval disjoint from E, so there exists some piecewise linear bump function φ such that $\varphi(x) = 1$ and $\varphi = 0$ outside of this interval. Then define $f_{x,E,f} = \varphi \circ f$. Let $\mathcal{G} = \{g_{x,E,f} : x, E, f \text{ as above}\}$. □

Definition 1.4. X is **completely regular** if it is T_1 and if for all closed $E \subseteq X$ and $x \in E^c$, there exists some $f \in C(X, I)$ such that $f(x) = 1$ and $f|_E = 0$.

This is sometimes called $T_{3\,1/2}$. So a T_1 space is completely regular if and only if $C(X, I)$ separates points and closed sets.

Definition 1.5. For $\mathcal{F} \subseteq C(X, I)$, the map associated to $\mathcal{F}$ is $e : X \to I^{\mathcal{F}} : x \mapsto (f(x))_{f \in \mathcal{F}}$.

We want to study when this is a homeomorphism.

Proposition 1.1. *Let $X, \mathcal{F}, e$ be as above.*

1. *e is continuous.*

2. *If $\mathcal{F}$ separates points, then e is injective.*

3. *If X is T_1 and $\mathcal{F}$ separates points and closed sets, then e is a homeomorphism $X \to e(X) \subseteq I^{\mathcal{F}}$.*

Proof. The first 2 mostly follow from the construction.

1. A canonical sub-base on $I^{\mathcal{F}}$ is sets of the form $\pi_f^{-1}[U] = \{(x_f)_{f\in\mathcal{F}} L x_f \in U\}$, where $U \subseteq [0,1]$ is open. Now $e^{-1}[\pi_f^{-1}[U]] = f^{-1}[U]$.

2. Let $x \neq y \in X$. Then there exists $f \in \mathcal{F}$ such that $(e_x)_f = f(x) \neq f(y) = (e(y))_f$. So $e(x) \neq e(y)$.

3. We must show that if U is open in X, then $e(U)$ is relatively open in $e(X)$. Pick $x \in U$. We will find an open subset V of $I^{\mathcal{F}}$ such that $e(x) \in V \cap e(X) \subseteq e(U)$; this implies that e^{-1} is continuous for the relative topology. Apply the assumption to x and $E = U^c$. Then there exists $f \in \mathcal{F}$ separating them, so $(e(x))_f \notin \overline{\pi_f(e[E])} = \overline{F(E)}$. Define $V = \{(y_g)_{g\in\mathcal{F}} : y_g \in I \setminus \overline{\pi_f(e[E])}\}$. This is open in $I^{\mathcal{F}}$. Then $e(x) \in V \cap e(X)$ by construction, and $V \cap e[E] = \varnothing$. So $V \cap e[X] \subseteq e[U]$. □

Corollary 1.1. *The following are equivalent:*

1. *X is completely regular.*

2. *X embeds into a cube.*

3. *X embeds into some compact Hausdorff space.*

Proof. (1) $\implies$ (2): Apply the proposition with $\mathcal{F} = C(X, I)$.

(2) $\implies$ (3): Cubes are compact Hausdorff spaces.

(3) $\implies$ (1): We just need that subsets of completely regular spaces are completely regular. Do this as an exercise. □

Corollary 1.2. *Any compact Hausdorff space is homeomorphic to a closed subset of a cube.*

Proof. X embeds into $e[X] \subseteq I^A$ for some A. Since X is compact, $e[X]$ is compact. I^A is Hausdorff, so $e[X]$ is closed. □

1.2 Compactification

In general, we can embed a completely regular space into a cube. Taking its closure, we get a compact, Hausdorff space.

Definition 1.6. A **compactification** of X is a pair (Y, φ), where Y is compact Hausdorff and φ is an embedding $X \to Y$ with $\overline{\varphi[X]} = Y$.

Example 1.1. $\mathbb{R} \to S^1$ is an embedding. If we add in the extra point, we get a **one-point compactification**.

Example 1.2. $\mathbb{R} \to [-1, 1]$ is an embedding. If we add the endpoints, we can get a two-point compactification.

In general, the compactification $X \to \overline{e[X]} \subseteq I^{(C(X,I)}$ is called the **Stone-Čech compactification**.

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "\varphi"'] & e[X] \arrow[d, dashed] \\ & Y \end{tikzcd}$$

1.3 Embeddings of compact spaces

Now let (X, ρ) be a compact metric space.

Lemma 1.1. *Compact metric spaces are separable.*

Proof. For all $n \in \mathbb{N}$, there exists a finite $S_n \subseteq X$ such that $\bigcup_{x \in S_n} B_{1/n}(x) = X$. Now $\bigcup_n S_n$ is countable and dense. $\square$

Corollary 1.3. $C(X)$ *is separable.*

Proof. Let $S \subseteq X$ be a countable dense subset. For $y \in S$, let $f_y(x) := \rho(y, x)$. Let $\mathcal{A}_{\mathbb{R}} := \{a_0 + \sum_{i=1}^{m} a_i f_{y_i} \cdots f_{y_i, m_i} : a_i \in \mathbb{R}, y_{i,j} \in S\}$. This is an algebra, it is nowhere vanishing, and it separates points : if $x \neq z$ in X, there exists $(y_n)_n \in S$ such that $y_n \to x$. So $f_{y_n}(x) \to 0$, and $f_{y-n}(z) \to \rho(x, z) \neq 0$. So $\overline{\mathcal{A}_{\mathbb{R}}}$ by the Stone-Weierstrass theorem, which means that $\overline{\mathcal{A}_{\mathbb{Q}}} = C(X)$. $\square$

Proposition 1.2. *Compact metric spaces embed into* $[0, 1]^{\mathbb{N}}$.

Proof. Let $\mathcal{A}$ be some countable dense subset of $C(X, I)$. Then A separates points and closed sets. So $[0, 1]^A \cong [0, 1]^{\mathbb{N}}$. $\square$

Remark 1.1. We can do this explicitly whenever X is separable. Let $(x_n)_n$ be dense in X. Let $e(x) := (\min\{\rho(x, x_n), 1\})_n \in [0, 1]^{\mathbb{N}}$. This is the embedding.

Theorem 1.1 (Urysohn's metrization theorem)**.** *Let X be 2nd countable. Then X is metrizable if and only if it is normal. Equivalently, X embeds into $[0,1]^{\mathbb{N}}$.*

Proof. Here is the idea for showing that normality implies that X embeds into $[0,1]^{\mathbb{N}}$. Let $\mathcal{E}$ be a countable base. Define the countabl collection $\mathcal{F}$ which separates U^c and V^c whenever $U, V \in \mathcal{E}$ and $U^c \cap V^c = \varnothing$. Now apply the embedding construction. $\square$